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On the Corrections to Scaling in Three-Dimensional Ising Models

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The leading correction-to-scaling amplitudes for the spin-1/2, nearest-neighbor sc, bcc, and fcc Ising models are considered with the particular aim of determining their signs. On the basis of previous two-variable series analyses by Chen, Fisher, and Nickel and renormalization group $\epsilon = 4 - d$ expansions, it is concluded that the correction amplitudes for the susceptibility, correlation length, specific heat, and spontaneous magnetization are *negative* for all three lattices. Thus, for example, the effective exponent $\gamma_{\mathrm{eff}}(T)$ asymptotically approaches the true susceptibility exponent γ from *above*. Other earlier and more recent high-temperature series and field-theoretic analyses are seen to be consistent with this result. However, the usual nonasymptotic, perturbative field-theoretic approaches are essentially committed to positive correction amplitudes. The question of the signs therefore relates directly to the applicability of these non-asymptotic field-theoretic calculations to three-dimensional Ising models as well as to different experimental systems.

KEY WORDS: Ising models; corrections to scaling; critical behavior; field theoretic calculations; epsilon expansions; series analysis.

1. INTRODUCTION

Nearly two decades ago, Wegner[1] pointed out that renormalization group theory implies that there are singular corrections to the leading power-law behavior of thermodynamic quantities near a critical point. For example, the susceptibility of a ferromagnet, in the absence of a bulk field, should be well described in the vicinity of its critical point by

$$\chi \approx C\,|t|^{-\gamma}(1 + a_\theta\,|t|^\theta + a_1 t + a_{2\theta}\,|t|^{2\theta} + a_2 t^2 + \cdots) \tag{1}$$

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where the reduced temperature $t = (T - T_c)/T_c$ measures the distance from the critical temperature. The leading exponent γ and the leading correction exponent θ are expected to be universal. Although the value of θ has been reliably calculated to be close to 0.5 by several different methods,[(2–10),3] the nonuniversal *correction amplitudes* a_θ remain somewhat elusive, even for relatively simple models such as the three-dimensional, nearest-neighbor, spin-$\frac{1}{2}$ ($s_i = \pm 1$) Ising models. In this note, we address, in particular, the question of the *signs* of these amplitudes for various thermodynamic functions, including the susceptibility, correlation length, specific heat, and spontaneous magnetization, in the spin-$\frac{1}{2}$ nearest-neighbor simple cubic (sc), body-centered cubic (bcc), and face-centered cubic (fcc) Ising models. (The nearest-neighbor and spin-$\frac{1}{2}$ restrictions will *not* be mentioned explicitly each time, but will be understood unless the context is clearly more general.) It transpires, as indicated below, that the signs of the leading corrections have implications for the applicability of certain field-theoretic techniques for describing the "preasymptotic" approach of the susceptibility, etc., to their limiting singular forms.

There have been several attempts to determine the correction-to-scaling amplitudes for Ising models, using series extrapolation. In 1975, Saul *et al.*[(2)] extrapolated high-temperature series for the susceptibility of the spin-S nearest-neighbor fcc Ising models. In addition, Camp and Van Dyke[(3)] extended and analyzed susceptibility series for the spin-S sc, bcc, and fcc models. Both groups found that the series were consistent with the value $\gamma = 1.250$ for all S, provided that a correction-to-scaling term of the form $a_\theta t^\theta$ with $\theta \simeq 0.5$ was included. Moreover, they concluded that the amplitude a_χ^+ for the leading correction term in the high-temperature susceptibility probably *vanished* for the spin-$\frac{1}{2}$ Ising model. In 1976, Camp *et al.*[(4)] concluded that correction-to-scaling terms were also undetectable in series for the correlation length and specific heat of the spin-$\frac{1}{2}$ Ising model. Since 1976, however, estimates of the exponent γ have dropped significantly.[(6,7,10–18)] The revised estimates result from sophisticated series analyses of the long series of Nickel[(11)] for a family of bcc lattice models in the same universality class as the Ising model,[(7,10–13)] as well as from renormalization group calculations.[(6,14–18)] In light of the revised values of γ and other exponents, it seems likely that the spin-$\frac{1}{2}$ estimates suggesting $a_\theta \simeq 0$ also need significant revision.[4]

[3] Including, in the case of Chen, Fisher and Nickel,[(10)] partial differential approximants: see Fisher and Chen.[(12)]

[4] A comprehensive review of the literature on this subject is beyond the scope of this note. Thus, we do not mention many further studies in the period following 1976; we have focused instead on more recent work based on the revised exponent estimates.

In 1980, Zinn-Justin[7] used a modified ratio method to extrapolate Nickel's high-temperature susceptibility and correlation length series for the general spin-S bcc Ising model.[11] He calculated universal *ratios* of the correction amplitudes, presumably gaining information on the individual amplitudes, but he does not mention their signs. More recently, George and Rehr[19] have analyzed high-temperature susceptibility, correlation length, and four-spin correlation series for the spin-S sc, bcc, and fcc Ising models. They find, via second-order differential approximants, that the correction-to-scaling amplitudes of the spin-$\frac{1}{2}$ Ising models are *negative* for the susceptibility and correlation length on all three lattices. Their results appear open to question, however, because they do not, in general, observe a consistent trend in a_θ with coordination number. For example, they report that the correlation length correction amplitudes are $a_\xi^+ = -0.177_5$, -0.084, and -0.094 for the sc, bcc, and fcc lattices, respectively, on the assumption $\gamma = 1.240$.[5] Although we know of no proof that there should be a monotonic trend with coordination number, we argue below that lack of such a trend is hard to accept. In most studies such uniform tendencies are the rule. Indeed, in a recent analysis[20] we found uniform, systematic trends with coordination number in the leading amplitudes for the susceptibility χ, correlation length ξ, specific heat C, and spontaneous magnetization M_0 of the sc, bcc, and fcc Ising lattices.

Since the results of George and Rehr are currently uncorroborated, it seems worthwhile to examine further the issue of correction-to-scaling amplitudes. The question of their signs is especially intriguing, as mentioned, because it relates directly to the applicability of the usual non-asymptotic field-theoretic calculations[21-23] to three-dimensional Ising models, and to experimental systems which they might approximate. This issue will be discussed in more detail below.

We argue here that the correction-to-scaling amplitudes a_θ for the susceptibility, correlation length, specific heat, and spontaneous magnetization are *negative* for the three-dimensional sc, bcc, and fcc spin-$\frac{1}{2}$ Ising models. We further show that earlier series extrapolation work[2-4] and recent series analyses[19,20] are consistent with this view and we explore briefly some of the implications of these results. Previous field-theoretic calculations,[21-23] on the other hand, are, as will be discussed below, effectively committed to *positive* values for a_θ.

[5] We understand from Dr. George that some of these results may be subject to reconsideration. We are indebted to him and Professor J. J. Rehr for correspondance.

2. SIGNS OF CORRECTION AMPLITUDES FOR THE BCC ISING MODEL

We begin by establishing that the correction amplitudes are negative for the bcc Ising model. The approach we follow is to imbed the pure bcc Ising model in a family of models which interpolate smoothly between the Gaussian model and the standard, spin-$\frac{1}{2}$ Ising model. Nickel[(11)] has developed long, two-variable series expansions for two such families of models—the double Gaussian (DG)[(10)] and Klauder (Kl)[(24)] models. The two variables used in the expansion are the high-temperature variable $x = J/k_B T$, where J is the exchange parameter, and the parameter y, which interpolates analytically from a pure Gaussian at $y = 0$ to a pure Ising model at $y = 1$. Both models have spin-weighting functions $W(s, y)$ for a continuous, scalar spin s located on each lattice site, which have the forms

$$\text{DG:} \qquad W(s, y) \propto b[e^{-b^2(s-\sqrt{y})^2} + e^{-b^2(s+\sqrt{y})^2}] \tag{2}$$

$$\text{Kl:} \qquad W(s, y) \propto b\,|s|^{y/(1-y)}\, e^{-b^2(s^2-1)} \tag{3}$$

where

$$b^2(y) = 1/2(1-y) \tag{4}$$

Nickel's three-dimensional bcc series to order x^{22} have been analyzed using special two-variable series extrapolation techniques by Chen, Fisher, and Nickel,[(10)] Fisher and Chen,[(12)] George and Rehr,[(9)] and Nickel and Rehr.[(13)] (In addition, Nickel calculated series for the two-dimensional square lattice; these were analyzed by Barma and Fisher,[(25)] who found vanishing amplitudes for the singular correction-to-scaling in the pure Ising limit $y = 1$, in accord with analytic information.[6]

Four of the results for the bcc lattice are pertinent here: (i) Chen *et al.*[(10)] showed that for nonvanishing y, both models belong to only one Ising-like universality class. (ii) Fisher and Chen[(12)] demonstrated that the leading correction amplitudes vanish linearly as $y - y_c$ near an Ising-like multicritical point $[y_c, x_c \equiv J/k_B T_c(y_c)]$, as illustrated in Fig. 1. (iii) Chen *et al.*[(10)] established that this multicritical point (C) lies *between* $y = 0$ (G) and $y = 1$ (I): see Fig. 1. Specifically, they found $y_c^{\mathrm{DG}} = 0.87 \pm 0.04$ and $y_c^{\mathrm{Kl}} = 0.81 \pm 0.06$ for the bcc models.[(10,12)] (iv) Chen *et al.*[(10)] found one and only one multicritical point in the range $0 < y < 1.8$ and the effective renor-

[6] Thus, for the two-dimensional case, Barma and Fisher[(25)] showed that the multicritical point y_c lies at $y = 1$, which is consistent with the absence of nonanalytic corrections to scaling in the two-dimensional Ising model. See also Aharony and Fisher.[(26)]

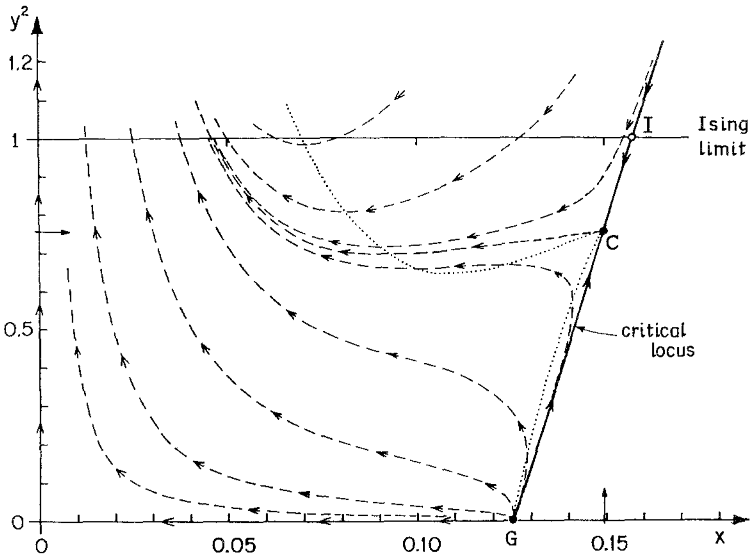

Fig. 1. Phase diagram in the $(x = J/k_B T, y)$ plane for the double-Gaussian model showing the critical locus $x_c(y)$ and the multicritical point C, which lies between the pure Gaussian point G and the pure Ising point I. The broken curves depict some partial differential flow trajectories which can be regarded as effective renormalization group flows. (After Chen *et al.*[10]).

malization group flows (see Fig. 1) established that this controlled the Ising-like behavior along the whole transition line for this range of y.

From these four results, we can infer a salient fact: since the three-dimensional, bcc Ising model and the Gaussian model lie on opposite sides of the multicritical point C, the correction amplitudes a_θ of the bcc Ising model must be *opposite in sign* to those near the Gaussian fixed point. In order to determine the signs of the latter corrections, we can follow several routes. The simplest relies on ϵ-expansions by Chang and Houghton.[27] (The second-order terms of their expressions for several universal correction amplitude *ratios* differ from those of Nicoll and Albright,[28] but their zeroth- and first-order terms are the same.[21]) To first order in ϵ, they find that the correction amplitues $a_\chi^\pm$, $a_\xi^\pm$, and $a_C^\pm$ for the susceptibility, correlation length, and specific heat above and below T_c, respectively, and a_M^- for the spontaneous magnetization, can be expressed as positive quantities multiplied by $-(u - u^*)/u^*$, where u is the coefficient of the ϕ^4 term in the field-theoretic Hamiltonian and $u^* > 0$ is its fixed point value. For u near the Gaussian fixed point, $0 \lesssim u < u^*$, their results therefore imply that the amplitudes $a_\chi^\pm$, $a_\xi^\pm$, $a_C^\pm$, and a_M^- are all positive.

Bagnuls and Bervillier's work using field theory in $d = 3$ dimensions[21] also supports this conclusion. They numerically calculate dimensionless functions corresponding to $\chi(T)$, $\xi(T)$, $C(T)$, and $M_0(T)$ and reduced temperature t for a set of values $\{u_p\}$ of the ϕ^4 coupling in the range $0 < u_p < u^*$. They find only positive correction amplitudes in this range ($u < u^*$), confirming the results of Chang and Houghton.[27]

Since the correction amplitudes for the spin-$\frac{1}{2}$ bcc Ising model are opposite in sign, we conclude that $a_\chi^\pm$, $a_\xi^\pm$, $a_C^\pm$, and a_M^- are *negative* for this Ising model. A corollary, based on the proportionality between the correction amplitudes and $-(u-u^*)/u^*$, is that the value of u corresponding to the bcc Ising model, say u^{bcc}, is greater than u^*; we will exploit this inequality later. Our overall conclusion is consistent with earlier results. In 1979, Nickel and Sharpe[29] observed that numerical evidence suggested $u > u^*$ for the spin-$\frac{1}{2}$ Ising models. In addition, George and Rehr[19] found that a_χ^+ and a_ξ^+ are negative for the bcc lattice. Our series extrapolation results[20] also support this conclusion for a_M^-. By fitting whole-range approximants for the spontaneous magnetization to the form

$$M_0(T) \simeq B\,|t|^\beta\,(1 + a_M^-\,|t|^{1/2}) \tag{5}$$

we found that $a_M^- \simeq -0.256$, -0.240, $-0.234 < 0$ for the sc, bcc, and fcc lattices (recall that θ is approximately 0.5).[7] We stress that although these values of a_M^- will provide good asymptotic representations of $M_0(T)$ for $t \gtrsim 10^{-5}$, they must not be regarded as definitive. It is, in fact, our belief that truly reliable quantitative estimates for the magnitudes (as opposed to the signs) of the correction-to-scaling amplitudes will only be obtained by analyzing continuous families of models with a line of critical points in the given universality class, like the DG and Klauder models, with the aid of two-variable techniques, such as partial differential approximants.[10,25] The theory of partial differential approximants is currently being extended so that effective calculations of the correction amplitudes can be performed.[30]

3. SIGNS OF CORRECTION AMPLITUDES FOR SC AND FCC ISING MODELS

Consider now the sc and fcc Ising models. In order to establish the signs of the correction amplitudes, we need to know whether the multicriti-

[7] We list here some of the more recent estimates of θ. Zinn-Justin[7] found, by a ratio analysis of the high-temperature bcc series of Nickel,[11] that $\theta = 0.52 \pm 0.07$. Chen *et al.*[10] examined the same series using partial differential approximants, and concluded that $\theta = 0.54 \pm 0.05$. George and Rehr[9] also extrapolated these series, but using a different partial differential approximant analysis, and found that the susceptibility series yielded $\theta = 0.52 \pm 0.03$ and the squared correlation length series yielded $\theta = 0.49 \pm 0.04$. On the field-theoretical front, Le Guillou and Zinn-Justin concluded $\theta = 0.498 \pm 0.020$ on the basis of loop expansions in $d = 3$[5,6] and $\theta = 0.504 \pm 0.026$ on the basis of ϵ-expansions.[17,18]

cal point y_c lies between $y=0$ and $y=1$ for the double Gaussian or Klauder models on the sc and fcc lattices, as it does for the bcc lattice. If so, the above arguments for the bcc Ising model apply, and the signs are negative for the sc and fcc Ising models as well. To date, no one has calculated two-variable series for these lattices. Here, we argue, on phenomenological grounds, that $y_c^{\text{sc}} < 1$, or equivalently, that $u^* < u^{\text{sc}}$. In fact, we suggest that $u^* < u^{\text{fcc}} < u^{\text{bcc}} < u^{\text{sc}}$ (and $0 < y_c^{\text{sc}} < y_c^{\text{bcc}} < y_c^{\text{fcc}} < 1$).

We may use the analysis of Seglar and Fisher,[31] who examined various crossovers, including that from Ising-like critical behavior to van der Waals behavior, which occurs when the range of interactions R_0 increases. If a denotes the lattice spacing, the infinite-range, infinitely-weak limit, or the Kac–van der Waals limit, corresponds to $R_0/a \to \infty$, or $p \equiv (a/R_0)^d \to 0$, where d is the spatial dimensionality. Seglar and Fisher demonstrated that this limit is equivalent to the Gaussian limit by starting with an initial spin Hamiltonian

$$\mathscr{H} = -\tfrac{1}{4} \sum_{j \neq i} J_{ij} |s_i - s_j|^2 - \tfrac{1}{2} D s_i^2 - \tfrac{1}{4} U s_i^4 \tag{6}$$

with interactions of the Kac form,

$$J_{ij} = J_0 (a/R_0)^d \, \varphi(R_{ij}/R_0) \tag{7}$$

By suitable spin and spatial rescalings, one can rewrite (6) as the standard ϕ^4 Hamiltonian

$$\mathscr{H}/k_B T = -\int d^dR \, [\tfrac{1}{2} (\nabla\phi)^2 + \tfrac{1}{2} r\phi^2 + \tfrac{1}{4} u\phi^4] \tag{8}$$

with a rescaled coupling constant given by[31]

$$u = (U k_B T / J_0^2) \, a^{-\epsilon} (a/R_0)^d \tag{9}$$

(where, as usual, $\epsilon = 4 - d$). Thus, when $p \equiv (a/R_0)^d$ approaches zero, the coupling u also vanishes, confirming that the Gaussian fixed point controls the infinite-range critical behavior. Consider now the relation between the coordination number q of a lattice and the range of interactions R_0. The mean field limit can be attained by letting either q or R_0 approach infinity while qJ or, as follows from (7), $J(R_0/a)^d$, remains finite.[32] Thus, as one would expect, $q \sim (R_0/a)^d$ and so the coupling constant u also obeys the asymptotic relation

$$u \sim q^{-1} \tag{10}$$

which implies $u^{\text{sc}} > u^{\text{bcc}} > u^{\text{fcc}}$.[8] But we have argued above that $u^{\text{bcc}} > u^*$;

[8] Note that the coordination number q for increasing range is not identical to the *nearest-neighbor* coordination number q_{nn}. Thus, the argument must be regarded as somewhat heuristic, unless one actually increases the spatial range *per se*, following the approach of Domb and Dalton.[33]

hence we conclude $u^{sc} > u^*$, so that the correction amplitudes are also negative for the susceptibility, correlation length, specific heat, and spontaneous magnetization of the sc Ising model. Indeed, on the basis of Chang and Houghton's results,[27] we would expect the amplitudes for the sc Ising model to be larger in magnitude than those for the bcc Ising model, since a_θ is proportional to $(u - u^*)/u^*$ and $(u^{sc} - u^*)/u^* > (u^{bcc} - u^*)/u^*$. George and Rehr[19] did, in fact, find that a_ξ^+ is largest in magnitude for the sc lattice (although, as mentioned, they do not find a consistent trend with coordination number). For the susceptibility, however, they report that $|a_\chi^+|$ is actually *smallest* for the sc lattice. In our series analyses,[20] we observed that a_M^- was most negative for the sc and least negative for the fcc lattice [see estimates immediately following (5) above], which agrees with expectation.

The argument given does not establish that the correction amplitudes are also negative for the fcc Ising model. Previous numerical results, however, suggest reasonably strongly that the signs should be the same for the fcc Ising model as for the bcc Ising model. First, as mentioned, George and Rehr[19] found negative correction amplitudes for χ and ξ on all three lattices for $T > T_c$. Second, we[20] obtained negative correction amplitudes for M_0 for all three lattices.[9] Finally, we[20] also found that leading amplitudes, for example, differed very little in numerical terms between the bcc and fcc lattices. For example, the low-temperature susceptibility amplitudes C^- were estimated to be 0.220 ± 0.004, $0.207_5 \pm 0.002$, and 0.205 ± 0.006 for the sc, bcc, and fcc lattices, respectively. Similarly, for the high-temperature correlation length amplitudes f_1^+, we found 0.495 ± 0.003, 0.4574 ± 0.0001, and 0.4462 ± 0.0005, respectively. In view of this similarity between the bcc and fcc lattices, it would be surprising if the correction amplitudes on the two lattices were to differ in sign. Thus, we believe that the correction amplitudes are negative for the susceptibility, correlation length, specific heat, and spontaneous magnetization on all three lattices.

This conclusion is in fact consistent with earlier analyses. As mentioned, Saul *et al.*,[2] Camp and Van Dyke,[3] and Camp *et al.*[4] concluded that the high-temperature correction amplitudes vanished for all three lattices, within uncertainties, if one has $\gamma = 1.25$. But it is now believed that $\gamma \simeq 1.239$. If we regard the higher estimate as the value of an *effective exponent* γ_{eff},[34] then we find

$$\gamma_{\text{eff}} \equiv -d \log \chi/dt = \gamma - \theta a_\theta \, |t|^\theta + \cdots \tag{11}$$

[9] In ref. 20, the spontaneous magnetization was the only function studied which showed clear, unbiased evidence of a confluent correction of the form $a_\theta |t|^\theta$, with $\theta \simeq 0.5$. Thus, Liu and Fisher did not estimate the correction amplitudes or attempt to determine their signs for the susceptibility, correlation length or specific heat.

where $\gamma \simeq 1.239$ and $\gamma_{\rm eff} \simeq 1.25$ for $t \simeq 10^{-3}$. Since $\gamma_{\rm eff} > \gamma$, this implies that a_θ is negative (provided higher-order corrections are unimportant for the accessible range of t). Similar arguments apply for ξ, C, and M_0.

The result $\gamma_{\rm eff} > \gamma$ may seem surprising, since the mean field and Gaussian value of γ is unity, which is *lower* than the Ising value. Indeed, the usual perturbative field-theoretic treatments start from the Gaussian fixed point or free-field limit and naturally lead *only* to an approach of the asymptotic exponent from the same side as the mean field value.[35] Values of $u > u^*$ are not accessible to the standard perturbative analysis. However, in the context of a specific model or real physical system, there is no general reason why $\gamma_{\rm eff}$ should vary monotonically from the mean field to the Ising values. This can be seen clearly by considering a Heisenberg model with Ising anisotropy as studied by Seglar and Fisher within the ϵ-expansion.[31] The asymptotic behavior must be Ising-like with, for $d = 3$, $\gamma \simeq 1.24$. Far from criticality, however, one sees a Heisenberg value of $\gamma(\simeq 1.40$ for $d = 3)$, which ultimately crosses over to the lower Ising value of γ from below. Seglar and Fisher also observed nonmonotonicity with effective exponents during various other crossovers. Thus, real systems could well show nonmonotonic variation in their effective exponents. This point was argued further by Fisher in relation to observations on micellar solutions.[10]

4. CONCLUSIONS

We consider now the implications of our results for nonasymptotic field-theoretic analyses, such as those of Bagnuls and Bervillier[21] and Dohm.[22,23] Both of these calculations rely on expansions in the coupling constant u performed by Nickel *et al.*[40] directly in $d = 3$ dimensions, and take into account correction terms of the form $a_{n\theta} |t|^{n\theta}$. As mentioned above, Bagnuls and Bervillier have numerically calculated dimensionless functions corresponding to the susceptibility and other functions, for values of u ranging from zero to u^*. According to Bagnuls and Bervillier, their functions should describe, via three adjustable parameters, experimentally measured (or theoretically calculated) functions in the "preasymptotic regime," where terms of order t or higher are negligible compared to the t^θ

[10] Fisher[36] suggested that long-range crossover might explain "nonuniversal" values of the exponent γ observed in micellar solutions by Degiorgio and co-workers.[37] In particular, he argued that the effective exponent might vary nonmonotonically from the expected Ising to mean-field behavior, possibly assuming values even lower than unity. Bagnuls and Bervillier[38] later showed that the variation of the effective exponent is actually monotonic within field theory in $d = 3$. More recently, Dietler and Cannell[39] have repeated the experiment and found no evidence of nonuniversal, or indeed, non-Ising, exponents.

term. The three adjustable parameters contain all the nonuniversal features except the actual sign of $u - u^*$, which is effectively assumed to be negative. Since their functions would seem to apply only to systems with $u < u^*$, they are not appropriate for describing the three-dimensional Ising models, or other systems with $u > u^*$, even in the preasymptotic regime. (However, their functions have been fitted successfully to the susceptibility, correlation length, and specific heat of xenon.[41]).

The nonasymptotic analysis of Dohm[22,23] also applies only to systems with $u < u^*$. However, his strategy differs from that of Bagnuls and Bervillier, in that the nonuniversal features are determined by fitting to *one* accurately known physical quantity, such as the susceptibility. The other desired quantities, for example, the specific heat, can then be obtained. Dohm has applied the strategy successfully to the superfluid transition in helium.[42] There appears to be a larger risk of error in this strategy, because it relies on having one well-known quantity which lies in the preasymptotic region. If, for a given system, the chosen quantity does not fully lie in this regime, then errors can be introduced in the fitting of non-universal features, and can be propagated through the calculation to affect the estimates of other physical functions.[11] An application of Dohm's method to Ising systems has been contemplated.[44] The approach would fit to the whole-range approximants for the correlation length which we presented[20] for the three-dimensional spin-$\frac{1}{2}$ Ising model. However, the approximants for the correlation length of the sc Ising model (which, as argued above, has the most negative correction terms) indicate that the effective exponent ν_{eff} approaches ν from *above*.[44] This is consistent with the negative value of the correction amplitude, and consequently with the inequality $u^{\text{sc}} > u^*$, as argued above.[12] Since the field-theoretic description used by Dohm, and by Bagnuls and Bervillier, implies positive correction amplitudes, it appears that its application to Ising models requires extensions of the field-theoretic analysis. (The need for such an extension has also been discussed by Bagnuls and Bervillier.[35),13])

[11] This difficulty can be avoid to some extent by determining the nonuniversal parameters by a simultaneous fit to two or three quantities. See, for example, ref. 43.

[12] The series analysis of the bcc and fcc correlation lengths in ref. 20, on the other hand, indicate that $\nu_{\text{eff}}(T)$ approaches ν from below for $t \gtrsim 3 \times 10^{-5}$. This could be interpreted as indicating a *positive correction amplitude*. However, we believe that the analytic corrections, $a_1 t$, etc. [see (1)] are responsible for the effect. The correction amplitudes, a_ξ^+, for the bcc and fcc models should, as argued, be smaller than for the sc lattice. Hence, they can be more easily overwhelmed by the analytic corrections, which appear already to play a role for rather small t.

[13] C. Bagnuls and C. Bervillier tell us that they have also performed field-theoretic-based calculations (unpublished) for $u > u^*$, which allow negative correction amplitudes.

An interesting question is whether $u < u^*$ for most experimental systems.[14] For the binary liquid mixture of carbon disulfide and nitromethane, for example, Greer[(46)] has analyzed coexistence curve data of Gopal *et al.*[(47)] and has concluded that the correction amplitude is *positive*; hence, it seems that $u < u^*$ for this mixture. This is also consistent with the fact that the effective exponent values $\gamma_{\rm eff}(T)$ observed for liquid–vapor compressibilities usually lie below 1.24 or 1.23.

In summary, we conclude that the signs of the correction amplitudes in the three-dimensional sc, bcc, and fcc spin-1/2 Ising models are all negative. The existing nonasymptotic field-theoretic calculations, on the other hand, entail positive correction amplitudes. In applications to nonasymptotic properties of experimental systems, such as liquid–vapor, binary liquid, or magnetic systems, the signs of the correction amplitudes may determine whether pure Ising model or field-theoretic calculations form a more useful basis of direct comparison.

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[14] B. G. Nickel has remarked to us that a similar situation occurs in the study of dilute polymer solutions. The interpenetration function, related to the second virial coefficient, has been calculated using both lattice self-avoiding random walk models and two-parameter models based on Gaussian chains. Although the two types of models yield leading correction terms which are opposite in sign, either sign can arise in an experiment. See especially Huber and Stockmayer.[(45)]

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